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Transverse eigenproblem of steady-state heat conduction for multi-dimensional two-layered slabs with automatic computation of eigenvalues

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Abstract

The paper analyses the transverse eigenvalue problem of nonconventional Sturm–Liouville type associated to the steady-state heat conduction in 3-D two-component slabs with imperfect thermal contact. In particular, it describes how the physical insight deriving from the transverse direction of six suitable 'homogeneous parallelepipeds' inherent to the considered two-layered parallelepiped is capable of providing useful and reasonably accurate information about the best bracketing bounds (lower and upper) for the roots (eigenvalues) of the transverse eigencondition. This information, in fact, enables one to establish starting points (initial guesses) for a root-finding iteration (e.g., Müller's method) so that convergence of the iteration may absolutely be guaranteed. Representative test examples are computed to illustrate the accuracy, reliability, and efficiency of the proposed fully automated solution algorithm.

1. Introduction

In a recent issue of this journal, de Monte [1] analysed the unsteady heat conduction in a two-dimensional two-layer isotropic-composite slab whose two slab-shaped regions were in perfect thermal contact. In particular, the composite plate was subjected to linear homogeneous boundary conditions of the third kind in the direction transverse to the layers. Then, in agreement with other research workers [2–8], he found that the solution was able to deal only with homogeneous boundary conditions of the first and second kind in the direction longitudinal to the layers.

Now, if the boundary value eigenproblem associated with the solution to the unsteady two-layer composite is split up into two one-dimensional eigenproblems, one across the layers and the other along the composite, the eigenproblem in the direction perpendicular to the layers reduces exactly to the one appropriate for steady-state heat conduction in a 3-D two-layered slab provided the two layers of the starting 2-D composite medium have the same thermal diffusivity. Of course, transverse eigenproblems of this type occur in steady-state heat conduction only when the two-layer body is having a nonhomogeneous boundary condition at a sidewall perpendicular to a direction longitudinal to the layers.

The objective of this work is to attentively analyse and then automatically compute the transverse eigenvalues characterising 3-D two-layer configurations with contact resistance for steady-state heat conduction. This is not a simple matter as the transverse eigenvalue problem is not of conventional Sturm–Liouville type because of the discontinuity of its coefficients [9,10]. To tell the true, some algorithms are available in the specialised literature [9,11–13]. But, as a matter of fact, they are not at all computation-effective since their execution requires either an endless succession of interactions between the user and the procedure itself [9,11] or an initial graphical representation of the eigencondition [12,13]. In both cases, however, the eigenvalue computation is not fully automatic and, consequently, is inherently time-consuming.

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Nomenclature

a_i, b, c	dimensions of the ith layer along x, y and z, respectively (Fig. 1)
Bi_i	Biot number for the ith layer: $h_i \cdot a_1/k_1$
$f(\beta_\xi)$	β_ξ-function on the LHS of the two-layer eigencondition (4)
$g(\nu_{e\xi})$	$\nu_{e\xi}$-function on the LHS of the single-layer eigencondition (19)
h_c	thermal contact conductance of the interface
h_i	heat transfer coefficient for the ith layer at the x-boundary side
k_i	thermal conductivity of the ith layer
L	entire dimension of the two-layered slab along x: $a_1 + a_2$
R_c	dimensionless thermal contact resistance of the interface: $k_1/(h_c \cdot a_1)$
$N_{IT}(\overline{N}_{IT})$	number of iterations (average)
x, y, z	rectangular space coordinates
$X'_{i,m}(\xi)$	mth ξ-eigenfunction corresponding to $\beta_{\xi,m}$ for the ith layer

Greek symbols

$\beta_{\xi,m}$	mth dimensionless transverse eigenvalue of the two-layer parallelepiped: $\lambda_{x,m} \cdot a_1$
γ	geometric ratio: a_2/a_1
ε_1, ε_2	first and second (stopping) convergence criteria of Müller's method
$\zeta_{\xi,m}$	mth initial guess for $\beta_{\xi,m}$
κ	transverse thermal conductivity ratio: k_2/k_1
$\lambda_{x,m}$	mth eigenvalue of the 3-D two-layered slab for x-direction
$\lambda_{jx,n}$	nth transverse eigenvalue of the jth 3-D homogeneous slab
$\nu_{j\xi,n}$	nth dimensionless transverse eigenvalue of the jth homogeneous parallelepiped: $\lambda_{jx,n} \cdot a_1$
ξ	dimensionless rectangular space coordinate: x/a_1

Subscripts

i	index (i.e. 1 or 2)
j	index for the homogeneous parallelepipeds of type a, b, c, d, e, f ($\kappa < 1$) and f ($\kappa > 1$) corresponding to Eqs. (13), (15), (17), (18), (20), (22) and (24), respectively, and inherent to the two-layered parallelepiped of Fig. 1
1	first layer and left sidewall ($x = -a_1$) of the two-layer parallelepiped of Fig. 1
2	second layer and right sidewall ($x = a_2$) of the 3-D two-layer slab of Fig. 1

Haji-Sheikh et al., instead, developed an efficient procedure to compute automatically and accurately these eigenvalues, as shown in the numerical examples of Refs. [8,14]. This procedure is based on a root-finding iteration (known as 'hybrid root finder') which is an application of a second-order modified Newton's and the bisection methods. The applicability of the hybrid root finder requires to supply the region (which is of remarkable concern!) where only one root is located. To this aim, Haji-Sheikh and Beck [8] as well as Aviles-Ramos et al. [14], noted (although within the limitations of only two numerical examples) that there exists only one eigenvalue between any two adjacent vertical asymptotes of the function corresponding to the eigencondition (this very useful result will also be proven in this work, but in a more general case). Their locations (known readily using explicit equations provided in Refs. [15,16]) were assumed by the same authors as lower and upper bounds for the transverse eigenvalues of the eigencondition in a free-asymptote format. Although the procedure proposed Haji-Sheikh et al. [8,14], is completely automatic, the width of the region (spacing between two neighbouring asymptotes) where the transverse eigenvalue is located can also be very large, and consequently the computational speed might considerably reduce when the hybrid root finder is applied.

In the present paper, a new, accurate, reliable and above all completely automated technique is developed by the author for calculating very computation-effective bounds (lower and upper) for the transverse eigenvalues of the eigencondition under consideration. It allows the interval widths to be considerably reduced (as shown in the text) with the great advantage to increase the computational speed when a root-finding iteration is applied.

The main feature of this technique is to be part-numerical and part-physical (i.e. not merely numerical!), as it calculates the best bracketing bounds for each eigenvalue across the layers (numerical aspect) by means of the physical insight deriving from six suitable 'homogeneous parallelepipeds' inherent to the considered two-layered parallelepiped (physical aspect). In particular, the transverse eigenvalues of these six 3-D homogeneous domains are determined using explicit equations [15,16].

Once the best bracketing bounds for the transverse eigenvalues have been derived, an algorithm based on Müller's method [17] is here used to find the roots of the transverse eigencondition. Contrarily to the classical root-finding iterations such as the secant, Newton-Raphson, bisection and hybrid root-finding methods, this algorithm requires the user to supply a vector containing the initial guesses for the transverse eigenvalues. As initial guesses we have assumed the best bracketing lower bounds. They make safe, reliable and well equipped the technique proposed for the fully automatic computation of the eigenvalues. In fact, they always guarantee convergence (i.e. roots of the eigencondition) of Müller's method accurately and with a sufficiently high computational speed.

Relevant and significant numerical examples related to different eigenvalue problems in the direction perpendicular to the layers have been selected to show the methodology, applicability and convergence behaviour of the eigenvalue-finding algorithm herein developed. The computations have been performed using Fortran syntax with double precision accuracy and, starting from the eigencondition, the early 20 eigenvalues with 15-decimal place accuracy have been obtained in less than 1 s using a standard desktop computer.

2. Formulation of the boundary value transverse eigenproblem

Consider a steady-state heat conduction problem in a 3-D two-layer isotropic-composite slab (Fig. 1) with thermal contact resistance and temperature-independent properties within each layer. The two-region composite parallelepiped is subject to linear homogeneous boundary conditions of the third kind in the direction transverse to the layers (i.e. x) and homogeneous boundary conditions of the first and (or) second kind in one of the two directions longitudinal to the layers, i.e. y or z. Finally, the 3-D heat-conducting two-layer body has a nonhomogeneous boundary condition in a very general way at a sidewall perpendicular to the remaining direction longitudinal to the layers, i.e. y or z, and has a boundary condition again homogeneous and of the first or second kind at the opposite parallel sidewall.

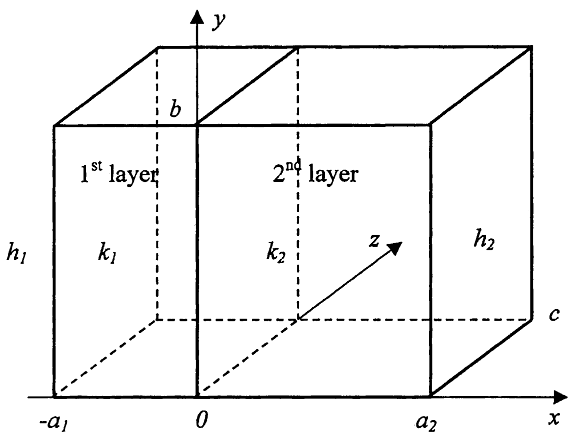

Fig. 1. Boundary conditions for the transverse eigenproblem of steady-state heat conduction within a two-layered parallelepiped.

From what was said, it is quite evident that the analytic solution to the problem here considered is able to deal only with homogeneous boundary conditions of the first and second kind in the directions longitudinal to the layers, with only one exception as pointed out above.

Then, the one-dimensional eigenvalue problem across the layers (i.e. in the x-direction) inherent to the solution of the considered heat conduction may be taken as

$$\frac{d^2X_i}{dx^2} + \lambda_x^2 \cdot X_i = 0 \tag{1a}$$

subject to the outer and inner boundary conditions, respectively

$$\mp k_i \cdot \left(\frac{dX_i}{dx}\right)_{x=\mp a_i} + h_i \cdot X_i(x=\mp a_i) = 0 \tag{1b}$$

$$X_1(x=0) + \frac{k_1}{h_c} \cdot \left(\frac{dX_1}{dx}\right)_{x=0} = X_2(x=0) \tag{1c}$$

$$k_1 \cdot \left(\frac{dX_1}{dx}\right)_{x=0} = k_2 \cdot \left(\frac{dX_2}{dx}\right)_{x=0} \tag{1d}$$

where the negative sign in Eq. (1b) is valid for $i = 1$, while the positive sign for $i = 2$. Additionally, the heat transfer coefficient h_i in Eq. (1b) may be chosen in such a way as to yield either Dirichlet, Neumann or Cauchy type boundary conditions.

It should be noted that the eigenproblem in the x-direction is completely independent of the eigenvalue problem in the direction longitudinal to the layers (i.e. along y or z) having homogeneous boundary conditions of the first and (or) second kind. Of course, the transverse eigenproblem posed by Eqs. (1) is also valid for steady-state heat conduction in 2-D two-layer isotropic-composite slabs having the same outer boundary conditions.

3. Solution to the transverse eigenvalue problem

It may be proven that the solutions for the space-variable transverse functions X_1 and X_2 can be expressed in the form

$$X_1(\xi) = c_\xi \cdot [\sin(\beta_\xi \cdot \xi) + \Pi_{1\xi}(\beta_\xi) \cdot \cos(\beta_\xi \cdot \xi)] \quad \xi \in [-1, 0] \tag{2a}$$

$$X_2(\xi) = c_\xi \cdot \frac{1}{\kappa} \cdot [\sin(\beta_\xi \cdot \xi) - \Pi_{2\xi}(\beta_\xi) \cdot \cos(\beta_\xi \cdot \xi)] \quad \xi \in [0, \gamma] \tag{2b}$$

where c_ξ is a constant (calculable through the nonhomogeneous boundary condition along the layers) and

$$\Pi_{1\xi}(\beta_\xi) = \frac{\beta_\xi + Bi_1 \cdot \tan(\beta_\xi)}{Bi_1 - \beta_\xi \cdot \tan(\beta_\xi)} \tag{3a}$$

$$\Pi_{2\xi}(\beta_\xi) = \frac{\beta_\xi + (Bi_2/\kappa)\cdot\tan(\gamma\cdot\beta_\xi)}{(Bi_2/\kappa) - \beta_\xi\cdot\tan(\gamma\cdot\beta_\xi)} \quad (3b)$$

The quantity $\beta_\xi = \lambda_x \cdot a_1$ appearing in Eqs. (2) and (3) is any root (dimensionless ξ-eigenvalue) other than zero of the following transcendental equation (transverse eigencondition)

$$\Pi_{1\xi}(\beta_\xi) + \frac{1}{\kappa}\cdot\Pi_{2\xi}(\beta_\xi) + \beta_\xi\cdot R_c = 0 \quad (4)$$

It may be proven (Section 4) that the roots $\beta_{\xi,m}$ of Eq. (4) are infinite, distinct and real according to Sturm–Liouville eigenproblems with stepwise functions [9,10]. Therefore, there are numerous space-variable functions having the forms (2a) and (2b), each corresponding to a consecutive value of the ξ-eigenvalues $\beta_{\xi,m}$ $(m = 0, 1, 2, 3, \ldots)$

$$X_{1,m}(\xi) = c_{\xi,m}\cdot X'_{1,m}(\xi) \quad \xi\in[-1,0] \quad (5a)$$

$$X_{2,m}(\xi) = c_{\xi,m}\cdot\frac{1}{\kappa}\cdot X'_{2,m}(\xi) \quad \xi\in[0,\gamma] \quad (5b)$$

The dimensionless functions $X'_{1,m}(\xi)$ and $X'_{2,m}(\xi)$, which appear in Eqs. (5), have been taken as the *ξ-eigenfunctions* corresponding to the ξ-eigenvalues $\beta_{\xi,m}$. They vanish for $\beta_{\xi,m=0} = 0$ ('trivial' eigenvalue) and satisfy the following orthogonality property

$$\int_{\xi=-1}^{0} X'_{1,m}\cdot X'_{1,n}\cdot d\xi + \frac{1}{\kappa}\cdot\int_{\xi=0}^{\gamma} X'_{2,m}\cdot X'_{2,n}\cdot d\xi = \begin{cases} 0 & \text{for } m\neq n \\ N^{+}_{\xi,m} & \text{for } m = n \end{cases} \quad (6)$$

It may be noted that the transverse eigenvalue equations listed before are also valid for transient heat conduction in multi-dimensional two-layer isotropic-composite slabs, provided the two layers have the same thermal diffusivity and are subject to homogeneous boundary conditions of the first and (or) second kind in the directions longitudinal to the layers, as shown in Ref. [1]. For this reason, Eqs. (2)–(6) may also be found in [1] where, however, a perfect thermal contact between the two layers was assumed ($R_c = 0$).

4. Analysis of the transverse eigenvalues

It may be proven that the transcendental equation (4) cannot have pure imaginary roots $\beta_\xi = \pm j\cdot q$, where q is a positive real number. In fact, if this were possible, the transverse eigencondition (4) would have the following expression

$$\frac{q + Bi_1\cdot\tanh(q)}{Bi_1 + q\cdot\tanh(q)} + \frac{1}{\kappa}\cdot\frac{q + (Bi_2/\kappa)\cdot\tanh(\gamma\cdot q)}{(Bi_2/\kappa) + q\cdot\tanh(\gamma\cdot q)} + q\cdot R_c = 0 \quad (7)$$

which is impossible as the three terms on the left-hand side are of the same sign (positive). It is interesting to note that, for only steady-state heat conduction in 2-D slabs (homogeneous or multi-layered), the above proof by absurd may also be carried out from a physical standpoint.

In fact, the temperature of the body can solely increase (or decrease) monotonically in the direction (e.g. y) where the 2-D body is having a nonhomogeneous boundary condition (this also occurs for a 3-D body, of course). The analytic solution to the problem confirms what was said, that is, the temperature increases (or decreases) as a hyperbolic function (i.e. hyperbolic sine or cosine) in the y-direction. Then, in view of the relations between hyperbolic and trigonometric functions and bearing in mind that for each 2-D layer $\beta_\psi = \beta_\xi$ (where β_ψ is the eigenvalue along the slab), there is no imaginary transverse eigenvalue because temperature cannot oscillates as space coordinate y increases (or decreases).

Additionally, it may also be proven that the ξ-eigencondition (4) cannot have a complex root of the form $\beta_\xi = p\pm j\cdot q$, where p and q are positive real numbers. In fact, if this were possible, we would have two conjugate roots $\beta_{\xi,m} = p + j\cdot q$ and $\beta_{\xi,n} = p - j\cdot q$, and consequently the ξ-eigenfunctions would become

$$X'_{1,r}(\xi) = \sin[(p\pm j\cdot q)\cdot\xi] + \Pi_{1\xi}(p\pm j\cdot q)\cdot\cos[(p\pm j\cdot q)\cdot\xi] \quad (r = m,n) \quad (8a)$$

$$X'_{2,r}(\xi) = \sin[(p\pm j\cdot q)\cdot\xi] - \Pi_{2\xi}(p\pm j\cdot q)\cdot\cos[(p\pm j\cdot q)\cdot\xi] \quad (r = m,n) \quad (8b)$$

where the positive sign is valid when $r = m$, while the negative sign when $r = n$. Applying the compound angle formulae for sines and cosines of the sum and difference of two angles and the relations between trigonometric and hyperbolic functions, after some algebraic steps the ξ-eigenfunctions (8) may be divided into their real and imaginary parts and rewritten as

$$X'_{1,r}(\xi) = R_1(\xi)\pm j\cdot S_1(\xi) \quad (r = m,n) \quad (9a)$$

$$X'_{2,r}(\xi) = R_2(\xi)\pm j\cdot S_2(\xi) \quad (r = m,n) \quad (9b)$$

Then, in view of the ξ-orthogonality property (6), we would have

$$\int_{\xi=-1}^{0}(R_1^2 + S_1^2)\cdot d\xi + \frac{1}{\kappa}\cdot\int_{\xi=0}^{\gamma}(R_2^2 + S_2^2)\cdot d\xi = 0 \quad (10)$$

which is impossible as both terms on the LHS are of the same sign (positive). Similarly to what was said for imaginary transverse eigenvalues, for only steady-state heat conduction in 2-D bodies (homogeneous or multi-layered), the above proof by absurd may still be performed from a physical standpoint. The only difference concerns the application of the compound angle formulae for hyperbolic sine and cosine of the sum and difference of two angles as the transverse eigenvalues are here complex and conjugate.

Therefore, the roots $\beta_{\xi,m}$ $(m = 1, 2, 3, \ldots)$ of the transcendental equation (4) are all real. In particular, the negative roots are equal in absolute value to the positive ones. To show that they also form a *monotonically increasing infinite series* in agreement with a Sturm–Liouville problem with discontinuous coefficients [9,10], the function $f(\beta_\xi)$ on the LHS of Eq. (4) has been analysed. It may be noted that this function has infinite vertical asymptotes, whose corresponding values of β_ξ may be determined as the solutions to the following two well-known transcendental equations

$$\beta_\xi \cdot \tan(\beta_\xi) - Bi_1 = 0 \tag{11a}$$

$$(\gamma \cdot \beta_\xi) \cdot \tan(\gamma \cdot \beta_\xi) - (Bi_2 \cdot \gamma/\kappa) = 0 \tag{11b}$$

Eq. (11a) is the same eigencondition for a single-layer slab with thickness a_1, with linear homogeneous boundary condition of the third kind at $x = -a_1$ (characterised by Bi_1) and homogeneous boundary condition of the second kind at $x = 0$. Similarly, Eq. (11b) is the same transcendental equation for a single-layer planar geometry with thickness a_2, with homogeneous boundary condition of the second kind at $x = 0$ and linear homogeneous boundary condition of the third kind at $x = a_2$ (characterised by $\widehat{B}i_2 = Bi_2 \cdot \gamma/\kappa$). The roots of Eqs. (11), and so the location of the vertical asymptotes of $f(\beta_\xi)$, may readily be determined by means of explicit approximate relations [15,16], in case followed by an iterative method (e.g. Müller's root-finding iteration [17]) to realise a high degree of accuracy (to 15 significant figures).

Once the vertical asymptotes have been calculated, it is an elementary matter to verify that, among them, the function $f(\beta_\xi)$ has a monotonically increasing trend as the first derivative $f'(\beta_\xi) > 0$. Therefore, the result $f'(\beta_\xi) > 0$ together with the existence of infinite vertical asymptotes states that (1) the function $f(\beta_\xi)$ crosses the β_ξ-axis infinite times, and (2) there exists only one transverse eigenvalue between any two adjacent asymptotes in accordance with what was established in Refs. [8,14], although there within the limitations of two numerical examples. Furthermore, the result $f'(\beta_\xi) > 0$ says that there are no repeated ξ-eigenvalues, i.e. the order of root multiplicity is equal to 1.

5. Bounds for the transverse eigenvalues

The vertical asymptotes of the $f(\beta_\xi)$ function may be assumed as lower and upper bounds for the transverse eigenvalues $\beta_{\xi,m}$ as suggested by Haji-Sheikh et al. [8,14], that is, as starting points for computing the eigenvalues $\beta_{\xi,m}$ of the free-asymptote format of Eq. (4) through a root-finding iteration [17]. Concerning this, Table 1 gives the first 20 intervals (lower and upper bounds—first two columns) where the first 20 dimensionless transverse eigenvalues $\beta_{\xi,m}$ of Eq. (4) are located when $Bi_1 = 1$, $Bi_2 = 2$, $\gamma = 2$, $\kappa = 100$ and $R_c = 1$.

However, behaving like this, the widths of the regions where the eigenvalues are located can also be very large (see Table 1) and, consequently, the computational speed might considerably reduce when the iterative method is applied.

For this reason, new more computation-effective bounds for the transverse eigenvalues $\beta_{\xi,m}$ are being sought in this work. To this purpose, it is interesting to note that the best bracketing bounds for each transverse eigenvalue $\beta_{\xi,m}$ may suitably be found by rewriting the eigencondition (4) in the following four forms

$$\Pi_{1\xi}(\beta_\xi) + (1/\kappa) \cdot [\Pi_{2\xi}(\beta_\xi) + \kappa \cdot \beta_\xi \cdot R_c] = 0 \tag{12a}$$

$$[\Pi_{1\xi}(\beta_\xi) + \beta_\xi \cdot R_c] + (1/\kappa) \cdot \Pi_{2\xi}(\beta_\xi) = 0 \tag{12b}$$

$$\kappa \cdot [\Pi_{1\xi}(\beta_\xi) + \Pi_{2\xi}(\beta_\xi)] + [(1-\kappa) \cdot \Pi_{2\xi}(\beta_\xi) + \kappa \cdot \beta_\xi \cdot R_c] = 0 \tag{12c}$$

$$[\Pi_{1\xi}(\beta_\xi) + \Pi_{2\xi}(\beta_\xi)] + [(\kappa-1) \cdot \Pi_{1\xi}(\beta_\xi) + \kappa \cdot \beta_\xi \cdot R_c] = 0 \tag{12d}$$

In particular, Eqs. (12c) and (12d) have been obtained adding and subtracting the quantities $\Pi_{2\xi}(\beta_\xi)$ and $\Pi_{1\xi}(\beta_\xi)/\kappa$ on the LHS of Eq. (4), respectively.

6. Bounds from the eigencondition (12a)

It may be proven that the first term on the LHS of Eq. (12a) vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_1 across the layers with k_1 as thermal conductivity, (2) linear homogeneous boundary condition of the third kind at $x = -a_1$ (characterised by h_1), and (3) homogeneous boundary condition of the first kind at $x = 0$. In fact, for this homogeneous parallelepiped (called afterwards 'HP-a' for the sake of brevity) the following well-known transverse eigenvalue equation applies

$$v_{a\xi} \cdot \cot(v_{a\xi}) + Bi_1 = 0 \tag{13}$$

whose roots $v_{a\xi,n}$ may be calculated by using explicit approximate equations [15,16]. Then, when the nth eigenvalue $v_{a\xi,n}$ of the homogeneous parallelepiped 'HP-a' falls in the mth region of the function $f(\beta_\xi)$ (with $m \neq n$ in general) where only one transverse eigenvalue (i.e. $\beta_{\xi,m}$) is located, it may be considered in every practical respect as bound for $\beta_{\xi,m}$. In particular, bearing in mind that $f(\beta_\xi)$ is a monotonically increasing function between any two neighbouring vertical asymptotes, the eigenvalue $v_{a\xi,n}$ will be assumed as a lower bound for $\beta_{\xi,m}$ (LB^m) if $f(v_{a\xi,n}) < 0$ as well as it will be considered as an

Table 1
First 20 intervals (lower and upper bounds-first two columns) where the first 20 transverse eigenvalues $\beta_{\xi,m}$ of Eq. (4) are located when $Bi_1 = 1$, $Bi_2 = 2$, $\gamma = 2$, $\kappa = 100$ and $R_c = 1$ (further bounds $v_{j\xi,n}$ (lower and upper) for $\beta_{\xi,m}$)

m	Lower bound for $\beta_{\xi,m}$	Upper bound for $\beta_{\xi,m}$	$v_{a\xi,n}$		$v_{b\xi,n}$		$v_{c\xi,n}$		$v_{d\xi,n}$		$v_{e\xi,n}$		$v_{f\xi,n}$ ($\kappa > 1$)	
1	***0.09933820***	0.86033358	***2.02875784***	UB^{3}	***0.12206881***	UB^{1}	0.86766214	LB^{2}	0.79792800	UB^{1}	0.41011093	UB^{1}	***1.30375641***	UB^{2}
2	0.86033358	1.57713659	***4.91318044***	UB^{6}	***1.58028789***	LB^{3}	3.42832310	LB^{5}	2.36043090	UB^{3}	***1.27422798***	LB^{2}	***3.67096465***	LB^{5}
3	1.57713659	3.14477249	***7.97866571***	UB^{9}	***3.14636002***	UB^{4}	6.43881551	LB^{8}	***3.92953562***	UB^{5}	2.23747458	UB^{3}	***6.58319903***	LB^{8}
4	***3.14477249***	3.42561845	***11.08553841***	UB^{12}	***4.71556992***	LB^{6}	9.53037238	LB^{11}	5.49960545	UB^{6}	3.24334012	UB^{4}	***9.63067871***	LB^{11}
5	3.42561845	4.71451007	***14.20743673***	UB^{15}	***6.28557172***	UB^{7}	12.64607310	LB^{14}	***7.06999789***	UB^{8}	4.26708513	UB^{5}	***12.72246903***	LB^{14}
6	4.71451007	6.28477644	***17.33637792***	UB^{18}	***7.85589103***	LB^{9}	15.77191639	LB^{17}	8.64053713	UB^{9}	5.29941475	UB^{6}	***15.83348124***	LB^{17}
7	***6.28477644***	6.43729817	20.46916740	–	***9.42636924***	UB^{10}	18.90293750	LB^{20}	***10.21115544***	UB^{11}	6.33641309	UB^{7}	***18.95444818***	LB^{20}
8	6.43729817	7.85525466	23.60428477	–	***10.99693830***	LB^{12}	22.03694958	–	11.78182122	UB^{12}	7.37620318	UB^{8}	22.08120952	–
9	7.85525466	9.42583887	26.74091601	–	***12.56756416***	UB^{13}	25.17284296	–	***13.35251770***	UB^{14}	8.41778635	UB^{9}	25.21163210	–
10	***9.42583887***	9.52933440	29.87858651	–	***14.13822789***	LB^{15}	28.30999565	–	14.92323520	UB^{15}	9.46058618	UB^{10}	28.34451266	–
11	9.52933440	10.99648366	33.01700103	–	***15.70891814***	UB^{16}	31.44803230	–	***16.49396771***	UB^{17}	10.50424802	UB^{11}	31.47912200	–
12	10.99648366	12.56716633	36.15596642	–	***17.27962767***	LB^{18}	34.58671310	–	18.06471132	UB^{18}	11.54854219	UB^{12}	34.61499291	–
13	***12.56716633***	12.64528722	39.29535098	–	***18.85035166***	UB^{19}	37.72587771	–	***19.63546337***	UB^{20}	12.59331364	UB^{13}	37.75181234	–
14	12.64528722	14.13787426	42.43506188	–	20.42108678	–	40.86541489	–	21.20622197	–	13.63845401	UB^{14}	40.88936281	–
15	14.13787426	15.70859986	45.57503180	–	21.99183065	–	44.00524505	–	22.77698578	–	14.68388535	UB^{15}	44.02748849	–
16	***15.70859986***	15.77128487	48.71521072	–	23.56258150	–	47.14530975	–	24.34775378	–	15.72955015	UB^{16}	47.16607489	–
17	15.77128487	17.27933832	51.85556073	–	25.13333805	–	50.28556512	–	25.91852522	–	16.77540509	UB^{17}	50.30503596	–
18	17.27933832	18.85008642	54.99605256	–	26.70409927	–	53.42597759	–	27.48929950	–	17.82141694	UB^{18}	53.44430583	–
19	***18.85008642***	18.90240995	58.13666324	–	28.27486439	–	56.56652101	–	29.06007616	–	18.86755975	UB^{19}	56.58383320	–
20	18.90240995	20.42084194	61.27737453	–	29.84563280	–	59.70717474	–	30.63085484	–	19.91381299	UB^{20}	59.72357751	–

LB^{m} = Lower bound for the mth transverse eigenvalue $\beta_{\xi,m}$. UB^{m} = Upper bound for the mth transverse eigenvalue $\beta_{\xi,m}$.

upper bound for $\beta_{\xi,m}$ (UB^m) if $f(v_{a\xi,n}) > 0$, as shown in Table 1.

Similarly, the second term between square brackets on the LHS of Eq. (12a) vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_2 across the layers with k_2 as thermal conductivity, and (2) linear homogeneous boundary conditions of the third kind at both $x = a_2$, characterised by h_2, and $x = 0$, characterised by h_1^* taken as

$$h_1^* = \frac{1}{R_c} \cdot \left(\frac{k_1}{a_1} \right) \tag{14}$$

In fact, for the homogeneous parallelepiped mentioned above (called afterwards 'HP-b') the following well-known transverse eigencondition is valid

$$\tan(\gamma \cdot v_{b\xi}) = \frac{(\gamma \cdot v_{b\xi}) \cdot [\gamma/(\kappa \cdot R_c) + Bi_2 \cdot \gamma/\kappa]}{(\gamma \cdot v_{b\xi})^2 - [\gamma/(\kappa \cdot R_c)] \cdot (Bi_2 \cdot \gamma/\kappa)} \tag{15}$$

whose roots $v_{b\xi,n}$ may easily be known using explicit relations [16]. Then, when the nth eigenvalue $v_{b\xi,n}$ of the homogeneous parallelepiped 'HP-b' falls in the mth interval of the function $f(\beta_\xi)$ (with $m \neq n$ in general) where only one transverse eigenvalue is located, it may be considered in every respect as bound for $\beta_{\xi,m}$, as shown in Table 1.

7. Bounds from the eigencondition (12b)

The first term between square brackets on the LHS of Eq. (12b) vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_1 across the layers with k_1 as thermal conductivity, and (2) linear homogeneous boundary conditions of Cauchy type at both $x = -a_1$ (characterised by h_1) and $x = 0$, characterised by h_2^* taken as

$$h_2^* = \frac{1}{\kappa \cdot R_c} \cdot \left(\frac{k_1}{a_1} \right) \tag{16}$$

In fact, for the above-mentioned homogeneous parallelepiped (called afterwards 'HP-c') the following well-known ξ-transcendental equation applies

$$\tan(v_{c\xi}) = \frac{v_{c\xi} \cdot [Bi_1 + 1/(\kappa \cdot R_c)]}{v_{c\xi}^2 - Bi_1 \cdot [1/(\kappa \cdot R_c)]} \tag{17}$$

whose eigenvalues $v_{c\xi,n}$ may easily be known using explicit equations [16]. Then, when the nth eigenvalue $v_{c\xi,n}$ of the homogeneous parallelepiped 'HP-c' falls in the mth region of the function $f(\beta_\xi)$ (with $m \neq n$ in general) where only one transverse eigenvalue (i.e. $\beta_{\xi,m}$) is located, it may be considered in every practical respect as bound for $\beta_{\xi,m}$. In particular, the eigenvalue $v_{c\xi,n}$ will be a lower bound for $\beta_{\xi,m}$ (LB^m) if $f(v_{c\xi,n}) < 0$ as well as it will be an upper bound for $\beta_{\xi,m}$ (UB^m) if $f(v_{c\xi,n}) > 0$, as indicated in Table 1.

As far as the second term on the LHS of Eq. (12b) is concerned, it may be demonstrated that it vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_2 across the layers with k_2 as thermal conductivity, (2) homogeneous boundary condition of the first kind at $x = 0$, and (3) linear homogeneous boundary condition of the third kind at $x = a_2$ (characterised by h_2). In fact, for this homogeneous parallelepiped (called afterwards 'HP-d') the following ξ-eigencondition is valid

$$(\gamma \cdot v_{d\xi}) \cdot \cot(\gamma \cdot v_{d\xi}) + (Bi_2 \cdot \gamma)/\kappa = 0 \tag{18}$$

whose roots $v_{d\xi,n}$ may readily be evaluated by using explicit approximate equations given in Refs. [15,16]. Then, when the nth eigenvalue $v_{d\xi,n}$ of the homogeneous parallelepiped 'HP-d' falls in the mth domain of the function $f(\beta_\xi)$ (with $m \neq n$ in general) where only one transverse eigenvalue is positioned, it may be considered as bound for $\beta_{\xi,m}$, as shown in Table 1.

8. Bounds from the eigencondition (12c)

The first term between square brackets on the LHS of Eq. (12c) vanishes for a 'homogeneous parallelepiped' having: (1) thickness $L = a_1 + a_2$ across the layers with k_1 as thermal conductivity, and (2) linear homogeneous boundary conditions of the third kind at both $x = -a_1$ (characterised by h_1) and $x = a_2$ (characterised by h_2/κ). In fact, for this homogeneous parallelepiped (called afterwards 'HP-e') the following transverse eigenvalue equation applies

$$\Pi_{1\xi}(v_{e\xi}) + \Pi_{2\xi}(v_{e\xi}) = 0 \tag{19}$$

Of course, only when $\kappa = 1$ and $R_c = 0$, the eigenvalue equation (4) reduces to the eigencondition (19) and we get in a straightforward manner $\beta_{\xi,n} \equiv v_{e\xi,n}$. In particular, the eigenvalues $v_{e\xi,n}$ of Eq. (19) may easily be evaluated as follows. In fact, applying the compound angle formula for tangents, after some algebraic steps Eq. (19) becomes

$$\tan[v_{e\xi} \cdot (1+\gamma)] = \frac{[v_{e\xi} \cdot (1+\gamma)] \cdot [Bi_1 \cdot (1+\gamma) + Bi_2 \cdot (1+\gamma)/\kappa]}{[v_{e\xi} \cdot (1+\gamma)]^2 - [Bi_1 \cdot (1+\gamma)] \cdot [Bi_2 \cdot (1+\gamma)/\kappa]} \tag{20}$$

whose roots $v_{e\xi,n}$ may readily be obtained by using explicit approximate equations given in [16].

The function $g(v_{e\xi})$ on the LHS of Eq. (19) crosses the $v_{e\xi}$-axis infinite times, and only one transverse eigenvalue $v_{e\xi,n}$ is located between any two neighbouring vertical asymptotes of $g(v_{e\xi})$. Additionally, a comparison between Eqs. (4) and (19) shows that the functions $f(\beta_\xi)$ and $g(v_{e\xi})$ have the same vertical asymptotes. Therefore, since the eigenvalues $v_{e\xi,n}$ of the homogeneous parallelepiped 'HP-e' always fall between any two adjacent asymptotes of the function $f(\beta_\xi)$ where only one transverse eigenvalue $\beta_{\xi,m}$ is located (therefore $m = n$), they may be considered in every practical respect as bounds for $\beta_{\xi,m}$. In particular, bearing in mind that $f(\beta_\xi)$ is a monotonically increasing function between any two neighbouring vertical asymptotes, $v_{e\xi,m}$ will be a lower bound for $\beta_{\xi,m}$ (LB^m) if $f(v_{e\xi,m}) < 0$ as well as it will be an upper bound for $\beta_{\xi,m}$ (UB^m) if $f(v_{e\xi,m}) > 0$, as shown in Table 1.

As far as the second term between square brackets on the LHS of Eq. (12c) is concerned, it vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_2 across the layers with k_2 as thermal conductivity, and (2) linear homogeneous boundary conditions of the third kind at both $x = a_2$, characterised by h_2, and $x = 0$, characterised by h_1^{**} taken as

$$h_1^{**} = \frac{1-\kappa}{R_c} \cdot \left(\frac{k_1}{a_1}\right) \tag{21}$$

provided κ be less than 1. In fact, for the homogeneous parallelepiped mentioned above (called afterwards 'HP-f') the following well-known transverse eigencondition is valid

$$\tan(\gamma \cdot v_{f\xi}) = \frac{(\gamma \cdot v_{f\xi}) \cdot [\gamma \cdot (1-\kappa)/(\kappa \cdot R_c) + Bi_2 \cdot \gamma/\kappa]}{(\gamma \cdot v_{f\xi})^2 - [\gamma \cdot (1-\kappa)/(\kappa \cdot R_c)] \cdot (Bi_2 \cdot \gamma/\kappa)} \quad \kappa < 1 \tag{22}$$

whose roots $v_{f\xi,n}$ may easily be known using explicit relations [16]. Then, when the nth eigenvalue $v_{f\xi,n}$ of the homogeneous parallelepiped 'HP-f' falls in the mth domain of the function $f(\beta_\xi)$ (with $m \neq n$ in general) where only one transverse eigenvalue (i.e. $\beta_{\xi,m}$) is positioned, it may be considered as bound for $\beta_{\xi,m}$.

9. Bounds from the eigencondition (12d)

It may be proven that the second term between square brackets on the LHS of Eq. (12d) vanishes for a 'homogeneous parallelepiped' having: (1) thickness a_1 across the layers with k_1 as thermal conductivity, and (2) linear homogeneous boundary conditions of Cauchy type at both $x = -a_1$ (characterised by h_1) and $x = 0$, characterised by h_2^{**} taken as

$$h_2^{**} = \frac{\kappa - 1}{\kappa \cdot R_c} \cdot \left(\frac{k_1}{a_1}\right) \tag{23}$$

provided $\kappa > 1$. In fact, for the above-mentioned homogeneous parallelepiped (called afterwards 'HP-f') the following eigenvalue ξ-equation is valid

$$\tan(v_{f\xi}) = \frac{v_{f\xi} \cdot [Bi_1 + (\kappa-1)/(\kappa \cdot R_c)]}{v_{f\xi}^2 - Bi_1 \cdot [(\kappa-1)/(\kappa \cdot R_c)]} \quad \kappa > 1 \tag{24}$$

whose roots $v_{f\xi,n}$ may easily be known using explicit relations [16]. Then, what was said for $v_{f\xi,n}$ in the previous section, may here be applied provided κ be greater than 1. The results are given in Table 1.

10. Best bracketing bounds for the transverse eigenvalues

From Table 1 we can establish in a straightforward manner the best bracketing bounds (indicated in bold-italic type) for either transverse eigenvalue $\beta_{\xi,m}$. Of course, a simple algorithm allows it to be done automatically.

It is very interesting to observe that the simplified cases expressed through the single-layer eigenconditions (13), (15), (17), (18), (20), (22) and (24), although do not represent the physical reality of the problem here under consideration, are however able to keep very useful physical insights in order to obtain some of the best bracketing bounds for each transverse eigenvalue in the more general case of the two-layered eigencondition (4).

For the sake of clarity, the best intervals for $\beta_{\xi,m}$ are given in Table 2. These intervals may be compared with the ones provided in Table 1 and obtained applying Haji-Sheikh and Beck's approach [8]. It is quite evident how the procedure here developed allows the interval widths to be considerably reduced. This implies a very high computational speed when an iterative method for computing the eigenvalues (e.g., the hybrid root finder) is applied. To this purpose, it should be noted that the computational time spent to calculate the bounds $v_{j\xi,n}$ (j = a, b, c, d, e, f) given in Table 1 is very low as explicit equations are used for their calculations.

10.1. Computation of the transverse eigenvalues

Müller's method [17] may successfully and efficiently be used to compute the roots of the transverse eigencondition (4). This method requires to supply a vector of length p containing the initial guesses $\zeta_{\xi,m}$ for the dimensionless transverse eigenvalues $\beta_{\xi,m}$ $(m = 1, 2, 3, \ldots, p)$.

As initial guesses we have assumed the best bracketing lower bounds for $\beta_{\xi,m}$. However, since some of these bounds may also be vertical asymptotes for the $f(\beta_\xi)$ function (see Table 1), to avoid numerical overflows the transverse eigencondition (4) has here been rewritten in the following free-asymptote format

$$c_1 \cdot \Psi_1(\beta_\xi) + c_2(\beta_\xi) \cdot \Psi_2(\beta_\xi) - c_3 \cdot \Psi_3(\beta_\xi) + c_4 \cdot [\Psi_4(\beta_\xi) - c_5(\beta_\xi) \cdot \Psi_5(\beta_\xi)] = 0 \tag{25}$$

where the functions c_i and Ψ_i $(i = 1, 2, \ldots, 5)$ are given in Appendix A.

The routine based on Müller's method has two convergence criteria. The first requires that the absolute

Table 2
Best bracketing bounds for the first 20 transverse eigenvalues $\beta_{\xi,m}$ of Eq. (4) when $Bi_1 = 1$, $Bi_2 = 2$, $\gamma = 2$, $\kappa = 100$ and $R_c = 1$

m	Lower bound	Upper bound	$\beta_{\xi,m}$	N_{IT}
1	0.09933820	0.12206881	0.107414505805431	7
2	1.27422798	1.30375641	1.303101042328689	8
3	1.58028789	2.02875784	1.582290323963676	7
4	3.14477249	3.14636002	3.145560971511817	6
5	3.67096465	3.92953562	3.673535876503403	6
6	4.71556992	4.91318044	4.715618980690687	6
7	6.28477644	6.28557172	6.285171548575676	7
8	6.58319903	7.06999789	6.584935695823281	7
9	7.85589103	7.97866571	7.855901343043847	6
10	9.42583887	9.42636924	9.426102379440740	7
11	9.63067871	10.21115544	9.631923981641890	7
12	10.99693830	11.08553841	10.996942035088200	6
13	12.56716633	12.56756416	12.567363996055720	7
14	12.72246903	13.35251770	12.723428925079390	7
15	14.13822789	14.20743673	14.138229644971080	6
16	15.70859986	15.70891814	15.708757998377030	7
17	15.83348124	16.49396771	15.834259248648630	7
18	17.27962767	17.33637792	17.279628626319100	6
19	18.85008642	18.85035166	18.850218208245540	7
20	18.95444818	19.63546337	18.955101207725830	7

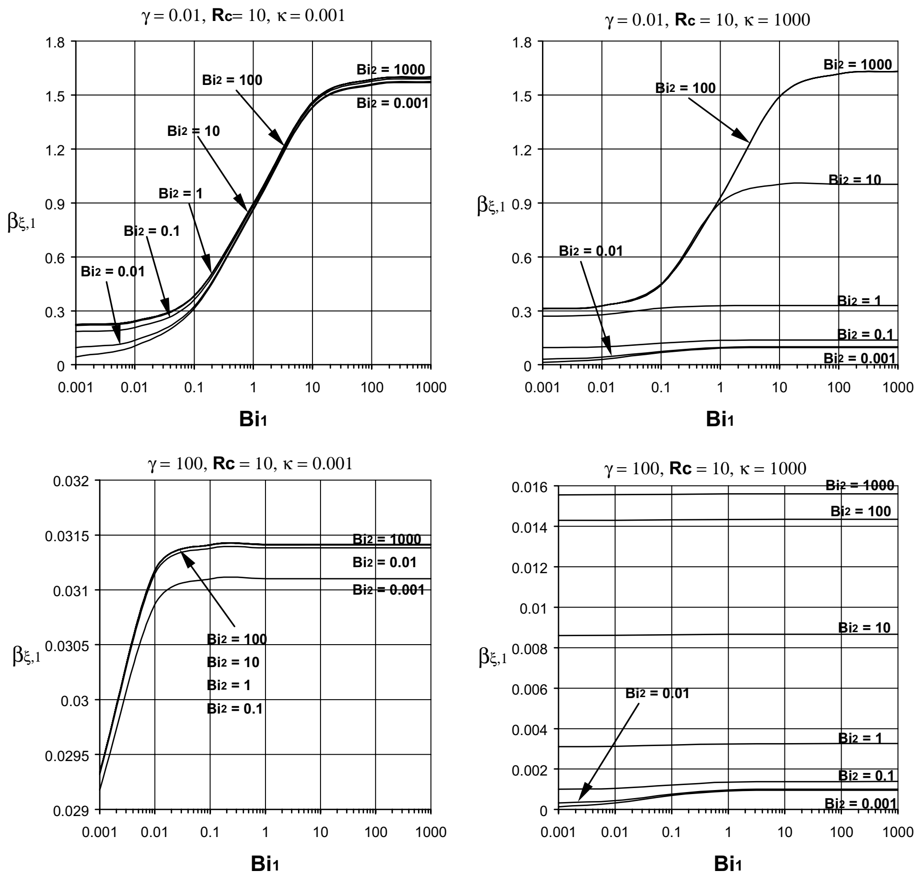

Fig. 2. $\beta_{\xi,1}$ versus Bi_1 with Bi_2 as a parameter for $\gamma = 10^{-2}$ (top) and $\gamma = 10^{2}$ (bottom), when $R_c = 10$, $\kappa = 10^{-3}$ (left side of figure) and $R_c = 10$, $\kappa = 10^{3}$ (right side of figure).

value of the β_ξ-function on the LHS of Eq. (25) at $\zeta^k_{\xi,m}$ be less than ε_1, where $\zeta^k_{\xi,m}$ is the kth approximation (i.e. kth iteration) to the mth initial guess $\zeta_{\xi,m}$. The second requires that the relative change of any two successive approximations $\zeta^k_{\xi,m}$ and $\zeta^{k+1}_{\xi,m}$ to the mth initial guess $\zeta_{\xi,m}$ be less than ε_2. "Convergence" is the satisfaction of either criterion. Then, starting from the best bracketing lower bounds (initial guesses $\zeta_{\xi,m}$) valid for the numerical example of Table 2, the early 20 eigenvalues $\beta_{\xi,m}$ have been computed and still provided in this table. It may be noticed that, setting $\varepsilon_1 = 1 \times 10^{-15}$ and $\varepsilon_2 = 1 \times 10^{-15}$, $\zeta_{\xi,m}$ lead to $\beta_{\xi,m}$ through 134 iterations ($\Rightarrow \overline{N}_{\mathrm{IT}} = 6.7$) with 15-decimal place accuracy.

At this stage, the reliability of the initial guesses proposed for reaching convergence of Müller method (i.e. $\beta_{\xi,m}$ of Eq. (25)) has to be proven for whichever value given to Bi_1, Bi_2, γ, κ and R_c. For this reason, wide ranges of these dimensionless groupings as well as several their combinations have been analysed carefully. The results are shown in Fig. 2 where, for the sake of brevity, only the first eigenvalue $\beta_{\xi,1}$ is plotted.

In particular, Fig. 2 gives the first dimensionless ξ-eigenvalue $\beta_{\xi,1}$ as a function of Bi_1 with Bi_2 as a parameter in correspondence of very heavy conditions, namely $\kappa = 10^{-3}$, $R_c = 10$ (left side of figure) and $\kappa = 10^3$, $R_c = 10$ (right side of figure). Concerning this, we recall that $\beta_{\xi,n} \equiv \nu_{e\xi,n}$ of Eq. (20) only when $\kappa = 1$ and $R_c = 0$. The computations have been performed setting $\varepsilon_1 = 1 \times 10^{-15}$ and $\varepsilon_2 = 1 \times 10^{-15}$, i.e. with 15-decimal place accuracy. Additionally, the geometric ratio γ was varied from 10^{-2} (top of figure) up to 10^2 (bottom of figure).

11. Conclusions

For many years Newton's method was the unchallenged method of computing transverse eigenvalues of nonconventional Sturm–Liouville problems in both steady-state and unsteady multi-layer multi-dimensional heat conduction, notwithstanding the time-consuming nature of the part-graphical, part-numerical work. The digital computer was the obvious tool for taking the labour out of the method whilst retaining—even enhancing—the insight. Perversely, in the present paper the computer has been fed with physically established starting points (i.e. initial guesses for computing the transverse eigenvalues through Müller's method) which were never part of the repertoire of respectable numerical analyses.

These points ensure that the convergence of the Müller root-finding iteration is always reached without any compromise in the 'natural' computation scheme here developed and, above all, without any suspect endless succession of interactions with the user. Therefore, a safe algorithm for the automatic computation of the transverse eigenvalues is ready to be implemented provided a free-asymptote format for the transverse eigencondition is used.

Appendix A

The functions c_i and Ψ_i $(i = 1, 2, \ldots, 5)$ appearing in the free-asymptote format (25) of the transverse eigencondition (4) are

$$\Psi_1(\beta_\xi) = Bi_1 \cdot Bi_2 \cdot \sin(\beta_\xi) \cdot \cos(\gamma \cdot \beta_\xi) - \kappa \cdot \beta_\xi^2 \cdot \sin(\gamma \cdot \beta_\xi) \cdot \cos(\beta_\xi)$$

$$\Psi_2(\beta_\xi) = \sin(\gamma \cdot \beta_\xi) \cdot \cos(\beta_\xi) - \sin(\beta_\xi) \cdot \cos(\gamma \cdot \beta_\xi)$$

$$\Psi_3(\beta_\xi) = \beta_\xi^2 \cdot [\kappa \cdot Bi_1 \cdot \sin(\gamma \cdot \beta_\xi) \cdot \cos(\beta_\xi) + Bi_2 \cdot \sin(\beta_\xi) \cdot \cos(\gamma \cdot \beta_\xi)]$$

$$\Psi_4(\beta_\xi) = \beta_\xi \cdot \cos(\gamma \cdot \beta_\xi) \cdot \cos(\beta_\xi)$$

$$\Psi_5(\beta_\xi) = \beta_\xi \cdot \sin(\gamma \cdot \beta_\xi) \cdot \sin(\beta_\xi)$$

$$c_1 = 1 \quad c_2(\beta_\xi) = \frac{Bi_1 \cdot Bi_2 + \kappa \cdot \beta_\xi^2}{\kappa + 1}$$

$$c_3 = \frac{R_c \cdot \kappa}{\kappa + 1} \quad c_4 = \frac{[Bi_1 \cdot (1 + Bi_2 \cdot R_c) + Bi_2] \cdot \kappa}{\kappa + 1}$$

$$c_5(\beta_\xi) = \frac{\kappa^2 \cdot Bi_1 + Bi_2 - (\kappa \cdot \beta_\xi)^2 \cdot R_c}{\kappa \cdot [Bi_1 \cdot (1 + Bi_2 \cdot R_c) + Bi_2]}$$

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